

# AN EFFECTIVE UNIVERSALITY THEOREM FOR THE RIEMANN ZETA FUNCTION

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ABSTRACT. Let  $0 < r < 1/4$ , and  $f$  be a non-vanishing continuous function in  $|z| \leq r$ , that is analytic in the interior. Voronin's universality theorem asserts that translates of the Riemann zeta function  $\zeta(3/4 + z + it)$  can approximate  $f$  uniformly in  $|z| < r$  to any given precision  $\varepsilon$ , and moreover that the set of such  $t \in [0, T]$  has measure at least  $c(\varepsilon)T$  for some  $c(\varepsilon) > 0$ , once  $T$  is large enough. This was refined by Bagchi who showed that the measure of such  $t \in [0, T]$  is  $(c(\varepsilon) + o(1))T$ , for all but at most countably many  $\varepsilon > 0$ . Using a completely different approach, we obtain the first effective version of Voronin's Theorem, by showing that in the rate of convergence one can save a small power of the logarithm of  $T$ . Our method is flexible, and can be generalized to other  $L$ -functions in the  $t$ -aspect, as well as to families of  $L$ -functions in the conductor aspect.

## 1. INTRODUCTION

In 1914 Fekete constructed a formal power series  $\sum_{n=1}^{\infty} a_n x^n$  with the following *universal* property: For any continuous function  $f$  on  $[-1, 1]$  (with  $f(0) = 0$ ) and given any  $\varepsilon > 0$  there exists an integer  $N > 0$  such that

$$\sup_{-1 \leq x \leq 1} \left| \sum_{n \leq N} a_n x^n - f(x) \right| < \varepsilon.$$

In the 1970's Voronin [14] discovered the remarkable fact that the Riemann zeta-function satisfies a similar universal property. He showed that for any  $r < \frac{1}{4}$ , any non-vanishing continuous function  $f$  in  $|z| \leq r$ , which is analytic in the interior, and for arbitrary  $\varepsilon > 0$ , there exists a  $T > 0$  such that

$$(1.1) \quad \max_{|z| \leq r} \left| \zeta\left(\frac{3}{4} + iT + z\right) - f(z) \right| < \varepsilon.$$

Voronin obtained a more quantitative description of this phenomena, stated below.

**Voronin's universality theorem.** *Let  $0 < r < \frac{1}{4}$  be a real number. Let  $f$  be a non-vanishing continuous function in  $|z| \leq r$ , that is analytic in the interior. Then, for any  $\varepsilon > 0$ ,*

$$(1.2) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \cdot \text{meas} \left\{ T \leq t \leq 2T : \max_{|z| \leq r} \left| \zeta\left(\frac{3}{4} + it + z\right) - f(z) \right| < \varepsilon \right\} > 0,$$

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where  $\text{meas}$  is Lebesgue's measure on  $\mathbb{R}$ .

There are several extensions of this theorem, for example to domains more general than compact discs (such as any compact set  $K$  contained in the strip  $1/2 < \text{Re}(s) < 1$  and with connected complement), or to more general  $L$ -functions. For a complete history of this subject, we refer the reader to [11].

The assumption that  $f(z) \neq 0$  is necessary: if  $f$  were allowed to vanish then an application of Rouché's theorem would produce at least  $\asymp T$  zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$  with  $\beta > \frac{1}{2} + \varepsilon$  and  $T \leq \gamma \leq 2T$ , contradicting the simplest zero-density theorems.

Subsequent work of Bagchi [1] clarified Voronin's universality theorem by setting it in the context of probability theory (see [7] for a streamlined proof). Viewing  $\zeta(\frac{3}{4} + it + z)$  with  $t \in [T, 2T]$  as a random variable  $X_T$  in the space of random analytic functions (i.e.  $X_T(z) = \zeta(\frac{3}{4} + iU_T + z)$  with  $U_T$  uniformly distributed in  $[T, 2T]$ ), Bagchi showed that as  $T \rightarrow \infty$  this sequence of random variables converges in law (in the space of random analytic functions) to a random Euler product,

$$\zeta(s, X) := \prod_p \left(1 - \frac{X(p)}{p^s}\right)^{-1}$$

with  $\{X(p)\}_p$  a sequence of independent random variables uniformly distributed on the unit circle (and with  $p$  running over prime numbers). This product converges almost surely for  $\text{Re}(s) > \frac{1}{2}$  and defines almost surely a holomorphic function in the half-plane  $\text{Re}(s) > \sigma_0$  for any  $\sigma_0 > \frac{1}{2}$  (see Section 2 below). The proof of Voronin's universality is then reduced to showing that the support of  $\zeta(s + 3/4, X)$  in the space of random analytic functions contains all non-vanishing analytic  $f : \{z : |z| < r\} \rightarrow \mathbb{C} \setminus \{0\}$ . Moreover it follows from Bagchi's work that the limit in Voronin's universality theorem exists for all but at most countably many  $\varepsilon > 0$ .

In this paper, we present an alternative approach to Bagchi's result using methods from hard analysis. As a result we obtain, for the first time, a rate of convergence in Voronin's universality theorem. We also give an explicit description for the limit in terms of the random model  $\zeta(s, X)$ .

**Theorem 1.1.** *Let  $0 < r < \frac{1}{4}$ . Let  $f$  be a non-vanishing continuous function on  $|z| \leq (r + 1/4)/2$  that is holomorphic in  $|z| < (r + 1/4)/2$ . Let  $\omega$  be a real-valued continuously differentiable function with compact support. Then, we have*

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \omega \left( \max_{|z| \leq r} |\zeta(\tfrac{3}{4} + it + z) - f(z)| \right) dt &= \mathbb{E} \left( \omega \left( \max_{|z| \leq r} |\zeta(\tfrac{3}{4} + z, X) - f(z)| \right) \right) \\ &\quad + O \left( (\log T)^{-\frac{(3/4-r)}{11} + o(1)} \right), \end{aligned}$$

where the constant in the  $O$  depends on  $f, \omega$  and  $r$ .

If the random variable  $Y_{r,f} = \max_{|z| \leq r} |\zeta(\frac{3}{4} + z, X) - f(z)|$  is absolutely continuous, then it follows from the proof of Theorem 1.1 that for any fixed  $\varepsilon > 0$  we have

$$\begin{aligned} & \frac{1}{T} \cdot \text{meas} \left\{ T \leq t \leq 2T : \max_{|z| \leq r} \left| \zeta\left(\frac{3}{4} + it + z\right) - f(z) \right| < \varepsilon \right\} \\ &= \mathbb{P} \left( \max_{|z| \leq r} \left| \zeta\left(\frac{3}{4} + z, X\right) - f(z) \right| < \varepsilon \right) + O \left( (\log T)^{-\frac{(3/4-r)}{11} + o(1)} \right). \end{aligned}$$

Unfortunately, we have not been able to even show that  $Y_{r,f}$  has no jump discontinuities. We conjecture the latter to be true, and one might even hope that  $Y_{r,f}$  is absolutely continuous.

A slight modification of the proof of Theorem 1.1 allows for more general domains than the disc  $|z| \leq r$ . Furthermore, if  $\omega \geq \mathbf{1}_{(0,\varepsilon)}$  (where  $\mathbf{1}_S$  is the indicator function of the set  $S$ ), then it follows from Voronin's universality theorem that the main term in Theorem 1.1 is positive. Explicit lower bounds for the limit in (1.2) (in terms of  $\varepsilon$ ) are contained in the papers of Good [3] and Garunkstis [2].

Our approach is flexible, and can be generalized to other  $L$ -functions in the  $t$ -aspect, as well as to “natural” families of  $L$ -functions in the conductor aspect. The only analytic ingredients that are needed are zero density estimates, and bounds on the coefficients of these  $L$ -functions (the so-called Ramanujan conjecture). In particular, the techniques of this paper can be used to obtain an effective version of a recent result of Kowalski [7], who proved an analogue of Voronin's universality theorem for families of  $L$ -functions attached to  $GL_2$  automorphic forms. In fact, using the zero-density estimates near 1 that are known for a very large class of  $L$ -functions (including those in the Selberg class by Kaczorowski and Perelli [6], and for families of  $L$ -functions attached to  $GL_n$  automorphic forms by Kowalski and Michel [8]), one can prove an analogue of Theorem 1.1 for these  $L$ -functions, where we replace  $3/4$  by some  $\sigma < 1$  (and  $r < 1 - \sigma$ ).

The main idea in the proof of Theorem 1.1 is to cover the boundary of the disc  $|z| \leq r$  with a union of a growing (with  $T$ ) number of discs, while maintaining a global control of the size of  $|\zeta'(s+z)|$  on  $|z| \leq r$ . It is enough to focus on the boundary of the disc thanks to the maximum modulus principle. The behavior of  $\zeta(s+z)$  with  $z$  localized to a shrinking disc is essentially governed by the behavior at a single point  $z = z_i$  in the disc. This allows us to reduce the problem to understanding the joint distribution of a growing number of shifts  $\log \zeta(s+z_i)$  with the  $z_i$  well-spaced, which can be understood by computing the moments of these shifts and using standard Fourier techniques.

It seems very difficult to obtain a rate of convergence which is better than logarithmic in Theorem 1.1. We have at present no understanding as to what the correct rate of convergence should be.

## 2. KEY INGREDIENTS AND DETAILED RESULTS

We first begin with stating certain important properties of the random model  $\zeta(s, X)$ . Let  $\{X(p)\}_p$  be a sequence of independent random variables uniformly distributed on the unit circle. Then we have

$$-\log \left( 1 - \frac{X(p)}{p^s} \right) = \frac{X(p)}{p^s} + h_X(p, s),$$

where the random series

$$(2.1) \quad \sum_p h_X(p, s),$$

converges absolutely and surely for  $\operatorname{Re}(s) > 1/2$ . Hence, it (almost surely) defines a holomorphic function in  $s$  in this half-plane. Moreover, since  $\mathbb{E}(X(p)) = 0$  and  $\mathbb{E}(|X(p)|^2) = 1$ , then it follows from Kolmogorov's three-series theorem that the series

$$(2.2) \quad \sum_p \frac{X(p)}{p^s}$$

is almost surely convergent for  $\operatorname{Re}(s) > 1/2$ . By well-known results on Dirichlet series, this shows that this series defines (almost surely) a holomorphic function on the half-plane  $\operatorname{Re}(s) > \sigma_0$ , for any  $\sigma_0 > 1/2$ . Thus, by taking the exponential of the sum of the random series in (2.1) and (2.2), it follows that  $\zeta(s, X)$  converges almost surely to a holomorphic function on the half-plane  $\operatorname{Re}(s) > \sigma_0$ , for any  $\sigma_0 > 1/2$ .

We extend the  $X(p)$  multiplicatively to all positive integers by setting  $X(1) = 1$  and  $X(n) := X(p_1)^{a_1} \cdots X(p_k)^{a_k}$ , if  $n = p_1^{a_1} \cdots p_k^{a_k}$ . Then we have

$$(2.3) \quad \mathbb{E} \left( X(n) \overline{X(m)} \right) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, for any complex number  $s$  with  $\operatorname{Re}(s) > 1/2$  we have almost surely that

$$\zeta(s, X) = \sum_{n=1}^{\infty} \frac{X(n)}{n^s}.$$

To compare the distribution of  $\zeta(s + it)$  to that of  $\zeta(s, X)$ , we define a probability measure on  $[T, 2T]$  in a standard way, by

$$\mathbb{P}_T(S) := \frac{1}{T} \operatorname{meas}(S), \text{ for any } S \subseteq [T, 2T].$$

The idea behind our proof of effective universality is to first reduce the problem to the discrete problem of controlling the distribution of many shifts  $\log \zeta(s_j + it)$  with all of the  $s_j$  contained in a compact set inside the strip  $\frac{1}{2} < \operatorname{Re}(s) < 1$ . One of the main ingredients in this reduction is the following result which allows us to control the maximum of the derivative of the Riemann zeta-function. This is proven in Section 4.

**Proposition 2.1.** *Let  $0 < r < 1/4$  be fixed. Then there exist positive constants  $b_1$ ,  $b_2$  and  $b_3$  (that depend only on  $r$ ) such that*

$$\mathbb{P}_T \left( \max_{|z| \leq r} |\zeta'(\tfrac{3}{4} + it + z)| > e^V \right) \ll \exp \left( -b_1 V^{\frac{1}{1-\sigma(r)}} (\log V)^{\frac{\sigma(r)}{1-\sigma(r)}} \right)$$

where  $\sigma(r) = \frac{3}{4} - r$ , uniformly for  $V$  in the range  $b_2 < V \leq b_3 (\log T)^{1-\sigma} / (\log \log T)$ .

We also prove an analogous result for the random model  $\zeta(s, X)$ , which holds for all sufficiently large  $V$ .

**Proposition 2.2.** *Let  $0 < r < 1/4$  be fixed and  $\sigma(r) = \frac{3}{4} - r$ . Then there exist positive constants  $b_1$  and  $b_2$  (that depend only on  $r$ ) such that for all  $V > b_2$  we have*

$$\mathbb{P} \left( \max_{|z| \leq r} |\zeta'(\tfrac{3}{4} + z, X)| > e^V \right) \ll \exp \left( -b_1 V^{\frac{1}{1-\sigma(r)}} (\log V)^{\frac{\sigma(r)}{1-\sigma(r)}} \right).$$

Once the reduction has been accomplished, it remains to understand the joint distribution of the shifts  $\{\log \zeta(s_1 + it), \log \zeta(s_2 + it), \dots, \log \zeta(s_J + it)\}$  with  $J \rightarrow \infty$  as  $T \rightarrow \infty$  at a certain rate, and  $s_1, \dots, s_J$  are complex numbers with  $\frac{1}{2} < \operatorname{Re}(s_j) < 1$  for all  $j \leq J$ . Heuristically, this should be well approximated by the joint distribution of the random variables  $\{\log \zeta(s_1, X), \log \zeta(s_2, X), \dots, \log \zeta(s_J, X)\}$ . In order to establish this fact (in a certain range of  $J$ ), we first prove, in Section 5, that the moments of the joint shifts  $\log \zeta(s_j + it)$  are very close to the corresponding ones of  $\log \zeta(s_j, X)$ , for  $j \leq J$ .

**Theorem 2.3.** *Fix  $1/2 < \sigma_0 < 1$ . Let  $s_1, s_2, \dots, s_k, r_1, \dots, r_\ell$  be complex numbers in the rectangle  $\sigma_0 < \operatorname{Re}(z) < 1$  and  $|\operatorname{Im}(z)| \leq T^{(\sigma_0 - 1/2)/4}$ . Then, there exist positive constants  $c_3, c_4, c_5$  and a set  $\mathcal{E}(T) \subset [T, 2T]$  of measure  $\ll T^{1-c_3}$ , such that if  $k, \ell \leq c_4 \log T / \log \log T$  then*

$$\begin{aligned} & \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \prod_{j=1}^k \log \zeta(s_j + it) \right) \left( \prod_{j=1}^\ell \log \overline{\zeta(r_j + it)} \right) dt \\ &= \mathbb{E} \left( \left( \prod_{j=1}^k \log \zeta(s_j, X) \right) \left( \prod_{j=1}^\ell \log \overline{\zeta(r_j, X)} \right) \right) + O(T^{-c_5}). \end{aligned}$$

Having obtained the moments we are in position to understand the characteristic function,

$$\Phi_T(\mathbf{u}, \mathbf{v}) := \frac{1}{T} \int_T^{2T} \exp \left( i \left( \sum_{j=1}^J (u_j \operatorname{Re} \log \zeta(s_j + it) + v_j \operatorname{Im} \log \zeta(s_j + it)) \right) \right) dt,$$

where  $\mathbf{u} = (u_1, \dots, u_J) \in \mathbb{R}^J$  and  $\mathbf{v} = (v_1, \dots, v_J) \in \mathbb{R}^J$ . We relate the above characteristic function to the characteristic function of the probabilistic model,

$$\Phi_{\text{rand}}(\mathbf{u}, \mathbf{v}) := \mathbb{E} \left( \exp \left( i \left( \sum_{j=1}^J (u_j \operatorname{Re} \log \zeta(s_j, X) + v_j \operatorname{Im} \log \zeta(s_j, X)) \right) \right) \right).$$

This is obtained in the following theorem, which we prove in Section 6.

**Theorem 2.4.** *Fix  $1/2 < \sigma < 1$ . Let  $T$  be large and  $J \leq (\log T)^\sigma$  be a positive integer. Let  $s_1, s_2, \dots, s_J$  be complex numbers such that  $\min(\operatorname{Re}(s_j)) = \sigma$  and  $\max(|\operatorname{Im}(s_j)|) < T^{(\sigma-1/2)/4}$ . Then, there exist positive constants  $c_1(\sigma), c_2(\sigma)$ , such that for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^J$  such that  $\max(|u_j|), \max(|v_j|) \leq c_1(\sigma)(\log T)^\sigma/J$  we have*

$$\Phi_T(\mathbf{u}, \mathbf{v}) = \Phi_{\text{rand}}(\mathbf{u}, \mathbf{v}) + O\left(\exp\left(-c_2(\sigma)\frac{\log T}{\log \log T}\right)\right).$$

Using this result, we can show that the joint distribution of the shifts  $\log \zeta(s_j + it)$  is very close to the corresponding joint distribution of the random variables  $\log \zeta(s_j, X)$ . The proof depends on Beurling-Selberg functions. To measure how close are these distributions, we introduce the discrepancy  $\mathcal{D}_T(s_1, \dots, s_J)$  defined as

$$\sup_{(\mathcal{R}_1, \dots, \mathcal{R}_J) \subset \mathbb{C}^J} \left| \mathbb{P}_T\left(\log \zeta(s_j + it) \in \mathcal{R}_j, \forall j \leq J\right) - \mathbb{P}\left(\log \zeta(s_j, X) \in \mathcal{R}_j, \forall j \leq J\right) \right|$$

where the supremum is taken over all  $(\mathcal{R}_1, \dots, \mathcal{R}_J) \subset \mathbb{C}^J$  and for each  $j = 1, \dots, J$  the set  $\mathcal{R}_j$  is a rectangle with sides parallel to the coordinate axes. Our next theorem, proven in Section 7, states a bound for the above discrepancy. This generalizes Theorem 1.1 of [10], which corresponds to the special case  $J = 1$ .

**Theorem 2.5.** *Let  $T$  be large,  $\frac{1}{2} < \sigma < 1$  and  $J \leq (\log T)^{\sigma/2}$  be a positive integer. Let  $s_1, s_2, \dots, s_J$  be complex numbers such that*

$$\frac{1}{2} < \sigma := \min_j(\operatorname{Re}(s_j)) \leq \max_j(\operatorname{Re}(s_j)) < 1 \quad \text{and} \quad \max_j(|\operatorname{Im}(s_j)|) < T^{(\sigma-1/2)/4}.$$

*Then, we have*

$$\mathcal{D}_T(s_1, \dots, s_J) \ll \frac{J^2}{(\log T)^\sigma}.$$

With all of the above tools in place we are ready to prove Theorem 1.1. This is accomplished in the next section.

### 3. EFFECTIVE UNIVERSALITY: PROOF OF THEOREM 1.1

In this section, we will prove Theorem 1.1 using the results described in Section 2. First, by the maximum modulus principle, the maximum of  $|\zeta(\frac{3}{4} + it + z) - f(z)|$  in the disc  $\{z : |z| \leq r\}$  must occur on its boundary  $\{z : |z| = r\}$ . Our idea consists of first covering the circle  $|z| = r$  with  $J$  discs of radius  $\varepsilon$  and centres  $z_j$ , where  $z_j \in \{z : |z| = r\}$  for all  $1 \leq j \leq J$ , and  $J \asymp 1/\varepsilon$ . We call each of the discs  $\mathcal{D}_j$ . Then, we observe that

$$(3.1) \quad \max_{j \leq J} |\zeta(\frac{3}{4} + it + z_j) - f(z_j)| \leq \max_{|z| \leq r} |\zeta(\frac{3}{4} + it + z) - f(z)| \leq \max_{j \leq J} \max_{z \in \mathcal{D}_j} |\zeta(\frac{3}{4} + it + z) - f(z)|.$$

Using Proposition 2.1, we shall prove that for all  $j \leq J$  (where  $J$  is a small power of  $\log T$ ) we have

$$\max_{z \in \mathcal{D}_j} |\zeta(\tfrac{3}{4} + it + z) - f(z)| \approx |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)|$$

for all  $t \in [T, 2T]$  except for a set of points  $t$  of very small measure. We will then deduce that the (weighted) distribution of  $\max_{|z| \leq r} |\zeta(\tfrac{3}{4} + it + z) - f(z)|$  is very close to the corresponding distribution of  $\max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)|$ , for  $t \in [T, 2T]$ . We will also establish an analogous result for the random model  $\zeta(s, X)$  along the same lines, by using Proposition 2.2 instead of Proposition 2.1. Therefore, to complete the proof of Theorem 1.1 we need to compare the distributions of  $\max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)|$  and  $\max_{j \leq J} |\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)|$ . Using Theorem 2.5 we prove

**Proposition 3.1.** *Let  $T$  be large,  $0 < r < 1/4$  and  $J \leq (\log T)^{(3/4-r)/7}$  be a positive integer. Let  $z_1, \dots, z_J$  be complex numbers such that  $|z_j| \leq r$ . Then we have*

$$\left| \mathbb{P}_T \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \leq u \right) - \mathbb{P} \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)| \leq u \right) \right| \ll_u \frac{(J \log \log T)^{6/5}}{(\log T)^{(3/4-r)/5}}.$$

*Proof.* Fix a positive real number  $u$ . Let  $\mathcal{A}_J(T)$  be the set of those  $t$  for which  $|\arg \zeta(\tfrac{3}{4} + it + z_j)| \leq \log \log T$  for every  $j \leq J$ . Since  $\operatorname{Re}(\tfrac{3}{4} + it + z_j) \geq \tfrac{3}{4} - r$  and  $\operatorname{Im}(\tfrac{3}{4} + it + z_j) = t + O(1)$ , then it follows from Theorem 1.1 and Remark 1 of [9] that for each  $j \leq J$  we have

$$(3.2) \quad \mathbb{P}_T (|\arg \zeta(\tfrac{3}{4} + it + z_j)| \geq \log \log T) \ll \exp \left( -(\log \log T)^{(\frac{1}{4}+r)^{-1}} \right) \ll \frac{1}{(\log T)^4}.$$

Therefore, we obtain

$$\mathbb{P}_T ([T, 2T] \setminus \mathcal{A}_J(T)) \leq \sum_{j=1}^J \mathbb{P}_T (|\arg \zeta(\tfrac{3}{4} + it + z_j)| \geq \log \log T) \ll \frac{J}{(\log T)^4} \ll \frac{1}{(\log T)^2},$$

and this implies that

$$(3.3) \quad \begin{aligned} \mathbb{P}_T \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \leq u \right) &= \mathbb{P}_T \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \leq u, t \in \mathcal{A}_J(T) \right) \\ &\quad + O \left( \frac{1}{(\log T)^2} \right). \end{aligned}$$

For each  $j \leq J$  consider the region

$$\mathcal{U}_j = \{z : |e^z - f(z_j)| \leq u, |\operatorname{Im}(z)| \leq \log \log T\}.$$

We cover  $\mathcal{U}_j$  with  $K \asymp \operatorname{area}(\mathcal{U}_j)/\varepsilon^2 \asymp (\log \log T)/\varepsilon^2$  squares  $\mathcal{R}_{j,k}$  with non-empty intersection with  $\mathcal{U}_j$ , and with sides of length  $\varepsilon = \varepsilon(T)$ , where  $\varepsilon$  is a small positive parameter to be chosen later. Let  $\mathcal{K}_j$  denote the set of  $k \in \{1, 2, \dots, K\}$  such that the

intersection of  $\mathcal{R}_{j,k}$  with the boundary of  $\mathcal{U}_j$  is empty and write  $\mathcal{K}_j^c$  for the relative complement of  $\mathcal{K}_j$  with respect to  $\{1, 2, \dots, K\}$ . Note that  $|\mathcal{K}_j^c| \asymp \log \log T/\varepsilon$ . By construction,

$$\left( \bigcup_{k \in \mathcal{K}_j} \mathcal{R}_{j,k} \right) \subset \mathcal{U}_j \subset \left( \bigcup_{k \leq K} \mathcal{R}_{j,k} \right).$$

Therefore (3.3) can be expressed as

$$\mathbb{P}_T \left( \forall j \leq J, \forall k \leq K : \log \zeta \left( \frac{3}{4} + it + z_j \right) \in \mathcal{R}_{j,k} \right) + \mathcal{E}_1$$

where by Theorem 2.5

$$\begin{aligned} \mathcal{E}_1 &\ll \sum_{j \leq J} \sum_{k \in \mathcal{K}_j^c} \mathbb{P}_T \left( \log \zeta \left( \frac{3}{4} + it + z_j \right) \in \mathcal{R}_{j,k} \right) \\ (3.4) \quad &\ll \sum_{j \leq J} \sum_{k \in \mathcal{K}_j^c} \left( \mathbb{P} \left( \log \zeta \left( \frac{3}{4} + z_j, X \right) \in \mathcal{R}_{j,k} \right) + \frac{1}{(\log T)^{3/4-r}} \right) \\ &\ll J \cdot \frac{\log \log T}{\varepsilon} \left( \varepsilon^2 + \frac{1}{(\log T)^{3/4-r}} \right), \end{aligned}$$

and in the last step we used the fact that  $\log \zeta(s, X)$  is an absolutely continuous random variable (see for example Jessen and Wintner [5]). We conclude that

$$\begin{aligned} (3.5) \quad \mathbb{P}_T \left( \max_{j \leq J} |\zeta \left( \frac{3}{4} + it + z_j \right) - f(z_j)| \leq u \right) &= \mathbb{P}_T \left( \forall j \leq J, \forall k \leq K : \log \zeta \left( \frac{3}{4} + it + z_j \right) \in \mathcal{R}_{j,k} \right) \\ &\quad + O \left( \varepsilon J \log \log T + \frac{J \log \log T}{\varepsilon (\log T)^{3/4-r}} \right). \end{aligned}$$

Additionally, it follows from Theorem 2.5 that the main term of this last estimate equals

$$(3.6) \quad \mathbb{P} \left( \forall j \leq J, \forall k \leq K : \log \zeta \left( \frac{3}{4} + z_j, X \right) \in \mathcal{R}_{j,k} \right) + O \left( \frac{J^2 (\log \log T)^2}{\varepsilon^4 (\log T)^{3/4-r}} \right).$$

We now repeat the exact same argument but for the random model  $\zeta(s, X)$  instead of the zeta function. In particular, instead of (3.2) we shall use that

$$\mathbb{P} \left( |\arg \zeta \left( \frac{3}{4} + z_j, X \right)| \geq \log \log T \right) \ll \exp \left( -(\log \log T)^{(\frac{1}{4}+r)^{-1}} \right) \ll \frac{1}{(\log T)^4},$$

which follows from Theorem 1.9 of [9]. Thus, similarly to (3.5) we obtain

$$\begin{aligned} \mathbb{P} \left( \forall j \leq J, \forall k \leq K : \log \zeta \left( \frac{3}{4} + z_j, X \right) \in \mathcal{R}_{j,k} \right) &= \mathbb{P} \left( \max_{j \leq J} |\zeta \left( \frac{3}{4} + z_j, X \right) - f(z_j)| \leq u \right) \\ &\quad + O \left( \varepsilon J \log \log T + \frac{J \log \log T}{\varepsilon (\log T)^{3/4-r}} \right). \end{aligned}$$



Combining the above estimate with (3.5) and (3.6) we conclude that

$$\begin{aligned} \mathbb{P}_T \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \leq u \right) &= \mathbb{P} \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)| \leq u \right) \\ &\quad + O \left( \frac{J^2 (\log \log T)^2}{\varepsilon^4 (\log T)^{3/4-r}} + \varepsilon J \log \log T \right). \end{aligned}$$

Finally, choosing

$$\varepsilon = \left( \frac{J \log \log T}{(\log T)^{3/4-r}} \right)^{1/5}$$

completes the proof.  $\square$

*Proof of Theorem 1.1.* We wish to estimate

$$(3.7) \quad \frac{1}{T} \int_T^{2T} \omega \left( \max_{|z| \leq r} |\zeta(\tfrac{3}{4} + it + z) - f(z)| \right) dt$$

with  $f$  an analytic non-vanishing function, and where  $\omega$  is a continuously differentiable function with compact support.

Recall that the maximum of  $|\zeta(\tfrac{3}{4} + it + z) - f(z)|$  on the disc  $\{z : |z| \leq r\}$  must occur on its boundary  $\{z : |z| = r\}$ , by the maximum modulus principle. Let  $\varepsilon \leq (1/4 - r)/4$  be a small positive parameter to be chosen later, and cover the circle  $|z| = r$  with  $J \asymp 1/\varepsilon$  discs  $\mathcal{D}_j$  of radius  $\varepsilon$  and centres  $z_j$ , where  $z_j \in \{z : |z| = r\}$  for all  $j \leq J$ .

Let  $\mathcal{S}_V(T)$  denote the set of those  $t \in [T, 2T]$  such that

$$\max_{|z| \leq (r+1/4)/2} |\zeta'(\tfrac{3}{4} + it + z)| \leq e^V$$

where  $V \leq \log \log T$  is a large parameter to be chosen later, and let  $L := \max_{|z| \leq (r+1/4)/2} |f'(z)|$ .

Then for  $t \in \mathcal{S}_V(T)$ , and for all  $z \in \mathcal{D}_j$  we have

$$\begin{aligned} (3.8) \quad & \left| \zeta(\tfrac{3}{4} + it + z) - f(z) - (\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)) \right| = \left| \int_{z_j}^z \zeta'(\tfrac{3}{4} + it + s) - f'(s) ds \right| \\ & \leq |z - z_j| \cdot \left( \max_{|z| \leq (r+1/4)/2} |\zeta'(\tfrac{3}{4} + it + z)| + L \right) \leq \varepsilon(e^V + L) \leq C\varepsilon e^V, \end{aligned}$$

for some large absolute constant  $C$ , depending at most on  $L$ . Define

$$\theta(t) := \max_{|z| \leq r} |\zeta(\tfrac{3}{4} + it + z) - f(z)| - \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)|.$$

Then, it follows from (3.1) and (3.8) that for all  $t \in \mathcal{S}_V(T)$  we have

$$(3.9) \quad 0 \leq \theta(t) \leq C\varepsilon e^V.$$

Therefore, using this estimate together with Proposition 2.1 and the fact that  $\omega$  is bounded, we deduce that (3.7) equals

$$\begin{aligned}
 (3.10) \quad & \frac{1}{T} \int_{t \in \mathcal{S}_V(T)} \omega \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| + \theta(t) \right) dt + O(e^{-V^2}) \\
 &= \frac{1}{T} \int_{t \in \mathcal{S}_V(T)} \omega \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \right) dt + O(|\mathcal{E}_2| + e^{-V^2}) \\
 &= \frac{1}{T} \int_T^{2T} \omega \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \right) dt + O(|\mathcal{E}_2| + e^{-V^2}),
 \end{aligned}$$

where

$$\mathcal{E}_2 = \frac{1}{T} \int_{t \in \mathcal{S}_V(T)} \int_0^{\theta(t)} \omega' \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| + x \right) dx \cdot dt \ll \varepsilon e^V,$$

using the fact that  $\omega'$  is bounded on  $\mathbb{R}$  together with (3.9).

Furthermore, observe that

$$\begin{aligned}
 (3.11) \quad & \frac{1}{T} \int_T^{2T} \omega \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \right) dt \\
 &= -\frac{1}{T} \int_T^{2T} \int_{\max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)|}^{\infty} \omega'(u) du \cdot dt \\
 &= -\int_0^{\infty} \omega'(u) \cdot \mathbb{P}_T \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \leq u \right) du.
 \end{aligned}$$

Since  $\omega$  has a compact support, then  $\omega'(u) = 0$  if  $u > A$  for some positive constant  $A$ .

Furthermore, it follows from Proposition 3.1 that for all  $0 \leq u \leq A$  we have

$$\begin{aligned}
 \mathbb{P}_T \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \leq u \right) &= \mathbb{P} \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)| \leq u \right) \\
 &\quad + O \left( \frac{(J \log \log T)^{6/5}}{(\log T)^{(3/4-r)/5}} \right).
 \end{aligned}$$

Inserting this estimate in (3.11) gives that

$$\begin{aligned}
 (3.12) \quad & \frac{1}{T} \int_T^{2T} \omega \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + it + z_j) - f(z_j)| \right) dt \\
 &= -\int_0^{\infty} \omega'(u) \cdot \mathbb{P} \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)| \leq u \right) du + O \left( \frac{(J \log \log T)^{6/5}}{(\log T)^{(3/4-r)/5}} \right) \\
 &= \mathbb{E} \left( \omega \left( \max_{j \leq J} |\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)| \right) \right) + O \left( \frac{(J \log \log T)^{6/5}}{(\log T)^{(3/4-r)/5}} \right).
 \end{aligned}$$

To finish the proof, we shall appeal to the same argument used to establish (3.10), in order to compare the (weighted) distributions of  $\max_{j \leq J} |\zeta(\tfrac{3}{4} + z_j, X) - f(z_j)|$  and  $\max_{|z| \leq r} |\zeta(\tfrac{3}{4} + z, X) - f(z)|$ . Let  $\mathcal{S}_V(X)$  denote the event corresponding to

$$\max_{|z| \leq (r+1/4)/2} |\zeta'(\tfrac{3}{4} + z, X)| \leq e^V,$$

and let  $\mathcal{S}_V^c(X)$  be its complement. Then, it follows from Proposition 2.2 that  $\mathbb{P}(\mathcal{S}_V^c(X)) \ll \exp(-V^2)$ . Moreover, similarly to (3.8) one can see that for all outcomes in  $\mathcal{S}_V(X)$  we have, for all  $z \in \mathcal{D}_j$

$$\left| \zeta\left(\frac{3}{4} + z, X\right) - f(z) - \left(\zeta\left(\frac{3}{4} + z_j, X\right) - f(z_j)\right) \right| = \left| \int_{z_j}^z \zeta'\left(\frac{3}{4} + s, X\right) - f'(s) ds \right| \ll \varepsilon e^V.$$

Thus, since the maximum of  $|\zeta(\frac{3}{4} + z, X) - f(z)|$  for  $|z| \leq r$  occurs (almost surely) on the boundary  $|z| = r$ , then following the argument leading to (3.10), we conclude that

$$\begin{aligned} & \mathbb{E} \left( \omega \left( \max_{|z| \leq r} |\zeta\left(\frac{3}{4} + z, X\right) - f(z)| \right) \right) \\ &= \mathbb{E} \left( \mathbf{1}_{\mathcal{S}_V(X)} \omega \left( \max_{|z| \leq r} |\zeta\left(\frac{3}{4} + z, X\right) - f(z)| \right) \right) + O(e^{-V^2}) \\ &= \mathbb{E} \left( \mathbf{1}_{\mathcal{S}_V(X)} \omega \left( \max_{j \leq J} |\zeta\left(\frac{3}{4} + z_j, X\right) - f(z_j)| \right) \right) + O(\varepsilon e^V + e^{-V^2}) \\ &= \mathbb{E} \left( \omega \left( \max_{j \leq J} |\zeta\left(\frac{3}{4} + z_j, X\right) - f(z_j)| \right) \right) + O(\varepsilon e^V + e^{-V^2}). \end{aligned}$$

Finally, combining this estimate with (3.10) and (3.12), and noting that  $J \asymp 1/\varepsilon$  we deduce that

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \omega \left( \max_{|z| \leq r} |\zeta\left(\frac{3}{4} + it + z\right) - f(z)| \right) dt &= \mathbb{E} \left( \omega \left( \max_{|z| \leq r} |\zeta\left(\frac{3}{4} + z, X\right) - f(z)| \right) \right) \\ &\quad + O \left( \varepsilon e^V + e^{-V^2} + O \left( \frac{(\log \log T)^{6/5}}{\varepsilon^{6/5} (\log T)^{(3/4-r)/5}} \right) \right). \end{aligned}$$

Choosing  $\varepsilon = (\log T)^{-(3/4-r)/11}$  and  $V = 2\sqrt{\log \log T}$  completes the proof.  $\square$

#### 4. CONTROLLING THE DERIVATIVES OF THE ZETA FUNCTION AND THE RANDOM MODEL: PROOF OF PROPOSITIONS 2.1 AND 2.2

By Cauchy's theorem we have

$$|\zeta'(\frac{3}{4} + it + z)| \leq \frac{1}{\delta} \max_{|s-z|=\delta} |\zeta(\frac{3}{4} + it + s)|,$$

and hence we get

$$(4.1) \quad \max_{|z| \leq r} |\zeta'(\frac{3}{4} + it + z)| \leq \frac{1}{\delta} \max_{|s| \leq r+\delta} |\zeta(\frac{3}{4} + it + s)|.$$

Therefore, it follows that

$$\begin{aligned} (4.2) \quad \mathbb{P}_T \left( \max_{|z| \leq r} |\zeta'(\frac{3}{4} + it + z)| > e^V \right) &\leq \mathbb{P}_T \left( \max_{|s| \leq r+\delta} |\zeta(\frac{3}{4} + it + s)| > \delta e^V \right) \\ &= \mathbb{P}_T \left( \max_{|s| \leq r+\delta} \log |\zeta(\frac{3}{4} + it + s)| > V + \log \delta \right). \end{aligned}$$

To bound the RHS we estimate large moments of  $\log \zeta(\frac{3}{4} + it + s)$ . This is accomplished by approximating  $\log \zeta(\frac{3}{4} + it + s)$  by a short Dirichlet polynomial, uniformly for all  $s$  in the disc  $\{|s| \leq r + \delta\}$ . Using zero density estimates and large sieve inequalities, we can show that such an approximation holds for all  $t \in [T, 2T]$ , except for an exceptional set of  $t$ 's with very small measure. We prove

**Lemma 4.1.** *Let  $0 < r < 1/4$  be fixed, and  $\delta = (1/4 - r)/4$ . Let  $y \leq \log T$  be a real number. There exists a set  $\mathcal{I}(T) \subset [T, 2T]$  with  $\text{meas}(\mathcal{I}(T)) \ll T^{1-\delta} y (\log T)^5$ , such that for all  $t \in [T, 2T] \setminus \mathcal{I}(T)$  and all  $|s| \leq r + \delta$  we have*

$$\log \zeta(\tfrac{3}{4} + it + s) = \sum_{n \leq y} \frac{\Lambda(n)}{n^{\frac{3}{4} + it + s} \log n} + O\left(\frac{(\log y)^2 \log T}{y^{(1/4-r)/2}}\right).$$

To prove this result, we need the following lemma from Granville and Soundararajan [4].

**Lemma 4.2** (Lemma 1 of [4]). *Let  $y \geq 2$  and  $|t| \geq y + 3$  be real numbers. Let  $1/2 \leq \sigma_0 < 1$  and suppose that the rectangle  $\{z : \sigma_0 < \text{Re}(z) \leq 1, |\text{Im}(z) - t| \leq y + 2\}$  is free of zeros of  $\zeta(z)$ . Then for any  $\sigma$  with  $\sigma_0 + 2/\log y < \sigma \leq 1$  we have*

$$\log \zeta(\sigma + it) = \sum_{n \leq y} \frac{\Lambda(n)}{n^{\sigma + it} \log n} + O\left(\log |t| \frac{(\log y)^2}{y^{\sigma - \sigma_0}}\right).$$

*Proof of Lemma 4.1.* Let  $\sigma_0 = 1/2 + \delta$ . For  $j = 1, 2$  let  $\mathcal{T}_j$  be the set of those  $t \in [T, 2T]$  for which the rectangle

$$\{z : \sigma_0 < \text{Re}(z) \leq 1, |\text{Im}(z) - t| < y + 1 + j\}$$

is free of zeros of  $\zeta(z)$ . Then, note that  $\mathcal{T}_2 \subseteq \mathcal{T}_1$ , and for all  $t \in \mathcal{T}_2$ , we have  $t + \text{Im}(s) \in \mathcal{T}_1$  for all  $|s| \leq r + \delta$ . Hence, by Lemma 4.2 we have

$$\log \zeta(\tfrac{3}{4} + it + s) = \sum_{n \leq y} \frac{\Lambda(n)}{n^{3/4 + it + s} \log n} + O\left(\frac{(\log y)^2 \log T}{y^{(1/4-r)/2}}\right),$$

for all  $t \in \mathcal{T}_2$  and all  $|s| \leq r + \delta$ . Let  $N(\sigma, T)$  be the number of zeros of  $\zeta(s)$  in the rectangle  $\sigma < \text{Re}(s) \leq 1$  and  $|\text{Im}(s)| \leq T$ . By the classical zero density estimate  $N(\sigma, T) \ll T^{3/2-\sigma} (\log T)^5$  (see for example Theorem 9.19 A of Titchmarsh [12]) we deduce that the measure of the complement of  $\mathcal{T}_2$  in  $[T, 2T]$  is  $\ll T^{1-\delta} y (\log T)^5$ .  $\square$

We also require a minor variant of Lemma 3.3 of [10], whose proof we will omit.

**Lemma 4.3.** *Fix  $1/2 < \sigma < 1$ , and let  $s$  be a complex number such that  $\text{Re}(s) = \sigma$ , and  $|\text{Im}(s)| \leq 1$ . Then, for any positive integer  $k \leq \log T / (3 \log y)$  we have*

$$\frac{1}{T} \int_T^{2T} \left| \sum_{n \leq y} \frac{\Lambda(n)}{n^{s+it} \log n} \right|^{2k} dt \ll \left( \frac{c_8 k^{1-\sigma}}{(\log k)^\sigma} \right)^{2k}$$

and

$$\mathbb{E} \left( |\log \zeta(s, X)|^{2k} \right) \ll \left( \frac{c_8 k^{1-\sigma}}{(\log k)^\sigma} \right)^{2k}$$

for some positive constant  $c_8$  that depends at most on  $\sigma$ .

*Proof of Proposition 2.1.* Let  $\eta = e^{-V/2}$ . Taking  $y = (\log T)^{5(1/4-r)^{-1}}$  in Lemma 4.1 gives for all  $t \in [T, 2T]$  except for a set with measure  $\ll T^{1-(1/4-r)/5}$  that

$$(4.3) \quad \log \zeta\left(\frac{3}{4} + it + s\right) = \sum_{n \leq y} \frac{\Lambda(n)}{n^{3/4+it+s} \log n} + O\left(\frac{1}{\log T}\right),$$

for all  $|s| \leq r + \eta$ . Furthermore, it follows from Cauchy's integral formula that

$$\left( \sum_{n \leq y} \frac{\Lambda(n)}{n^{3/4+it+s} \log n} \right)^{2k} = \frac{1}{2\pi i} \int_{|z|=r+2\eta} \left( \sum_{n \leq y} \frac{\Lambda(n)}{n^{3/4+it+z} \log n} \right)^{2k} \frac{dz}{z-s}.$$

Applying Lemma 4.3 we get that

$$(4.4) \quad \begin{aligned} \frac{1}{T} \int_T^{2T} \left( \max_{|s| \leq r+\eta} \left| \sum_{n \leq y} \frac{\Lambda(n)}{n^{3/4+it+s} \log n} \right| \right)^{2k} dt &\ll \frac{1}{\eta} \int_{|z|=r+2\eta} \frac{1}{T} \int_T^{2T} \left| \sum_{n \leq y} \frac{\Lambda(n)}{n^{3/4+it+z} \log n} \right|^{2k} dt |dz| \\ &\ll e^{V/2} \left( c_8(r) \frac{k^{1-\sigma'(r)}}{(\log k)^{\sigma'(r)}} \right)^{2k} \end{aligned}$$

where  $\sigma'(r) = \frac{3}{4} - r - 2\eta$ , and  $k \leq c_9 \log T / \log \log T$ , for some sufficiently small constant  $c_9 > 0$ . We now choose  $k = \lfloor c_6(r) V^{\frac{1}{1-\sigma(r)}} (\log V)^{\frac{\sigma(r)}{1-\sigma(r)}} \rfloor$  (so that  $k^{\sigma'(r)} \asymp k^{\sigma(r)}$ ) where  $c_6(r)$  is a sufficiently small absolute constant. Using (4.2) and (4.3) along with Chebyshev's inequality and the above estimate we conclude that there exists  $c_7(r) > 0$  such that

$$\begin{aligned} \mathbb{P}_T \left( \max_{|z| \leq r} |\zeta'(\tfrac{3}{4} + it + z)| > e^V \right) &\ll \mathbb{P}_T \left( \max_{|s| \leq r+\eta} \left| \sum_{n \leq y} \frac{\Lambda(n)}{n^{3/4+it+s} \log n} \right| > \frac{V}{4} \right) + T^{-(1/4-r)/5} \\ &\ll e^{V/2} \left( \frac{4}{V} \cdot c_8(r) \frac{k^{1-\sigma'(r)}}{(\log k)^{\sigma'(r)}} \right)^{2k} + T^{-(1/4-r)/5} \\ &\ll \exp \left( -c_6 V^{\frac{1}{1-\sigma(r)}} (\log V)^{\frac{\sigma(r)}{1-\sigma(r)}} \right) \end{aligned}$$

for  $V \leq c_7(\log T)^{1-\sigma(r)} / \log \log T$ .

□

We now prove Proposition 2.2 along the same lines. The proof is in fact easier than in the zeta function case, since we can compute the moments of  $\log \zeta(s, X)$ , for any  $s$  with  $\operatorname{Re}(s) > 1/2$ .

*Proof of Proposition 2.2.* Let  $\eta = e^{-V/2}$ . Since  $\zeta(\frac{3}{4} + s, X)$  is almost surely analytic in  $|s| \leq r + 2\eta$ , then by Cauchy's estimate we have almost surely that

$$\max_{|z| \leq r} |\zeta'(\frac{3}{4} + z, X)| \leq \frac{1}{\eta} \max_{|s| \leq r + \eta} |\zeta(\frac{3}{4} + s, X)|.$$

Therefore, we obtain

$$(4.5) \quad \begin{aligned} \mathbb{P} \left( \max_{|z| \leq r} |\zeta'(\frac{3}{4} + z, X)| > e^V \right) &\leq \mathbb{P} \left( \max_{|s| \leq r + \eta} |\zeta(\frac{3}{4} + s, X)| > \eta e^V \right) \\ &\leq \mathbb{P} \left( \max_{|s| \leq r + \eta} |\log \zeta(\frac{3}{4} + s, X)| > \frac{V}{2} \right). \end{aligned}$$

Let  $k$  be a positive integer. By (2.2)  $\log \zeta(\frac{3}{4} + s, X)$  converges almost surely to a holomorphic function in  $|s| \leq r + 2\eta$ . Using Cauchy's integral formula as in (4.4), we obtain almost surely that

$$\left( \max_{|s| \leq r + \eta} |\log \zeta(\frac{3}{4} + s, X)| \right)^{2k} \ll \frac{1}{\eta} \int_{|z|=r+2\eta} |\log \zeta(\frac{3}{4} + z, X)|^{2k} \cdot |dz|.$$

Hence, applying Lemma 4.3 we get

$$(4.6) \quad \begin{aligned} \mathbb{P} \left( \max_{|s| \leq r + \eta} |\log \zeta(\frac{3}{4} + s, X)| > V/2 \right) &\leq \left( \frac{2}{V} \right)^{2k} \cdot \mathbb{E} \left( \left( \max_{|s| \leq r + \eta} |\zeta(\frac{3}{4} + s, X)| \right)^{2k} \right) \\ &\ll \left( \frac{2}{V} \right)^{2k} e^{V/2} \int_{|z|=r+2\eta} \mathbb{E} (|\log \zeta(\frac{3}{4} + z, X)|^{2k}) \cdot |dz| \\ &\ll e^{V/2} \left( \frac{2c_8(r)k^{1-\sigma'(r)}}{V(\log k)^{\sigma'(r)}} \right)^{2k}, \end{aligned}$$

where  $\sigma'(r) = \frac{3}{4} - r - 2\eta$ . Let  $\sigma(r) = \frac{3}{4} - r$  and take  $k = \lfloor c_6 V^{\frac{1}{1-\sigma(r)}} (\log V)^{\frac{\sigma(r)}{1-\sigma(r)}} \rfloor$ , where  $c_6$  is sufficiently small (note that  $k^{\sigma'(r)} \asymp k^{\sigma(r)}$ ), then apply (4.6) to complete the proof.  $\square$

## 5. MOMENTS OF JOINT SHIFTS OF $\log \zeta(s)$ : PROOF OF THEOREM 2.3

The proof of Theorem 2.3 splits into two parts. In the first part we derive an approximation to

$$\prod_{j=1}^k \log \zeta(s_j + it)$$

by a short Dirichlet polynomial. In the second part we compute the resulting mean-values and obtain Theorem 2.3.

**5.1. Approximating  $\prod_{j=1}^k \log \zeta(s_j + it)$  by short Dirichlet polynomials.** Fix  $1/2 < \sigma_0 < 1$ , and let  $\delta := \sigma_0 - 1/2$ . Let  $k \leq \log T$  be a positive integer and  $s_1, s_2, \dots, s_k$  be complex numbers (not necessarily distinct) in the rectangle  $\{z : \sigma_0 \leq \operatorname{Re}(z) < 1, \text{ and } |\operatorname{Im}(z)| \leq T^{\delta/4}\}$ . We let  $\mathbf{s} = (s_1, \dots, s_k)$ , and define

$$F_{\mathbf{s}}(n) = \sum_{\substack{n_1, n_2, \dots, n_k \geq 2 \\ n_1 n_2 \cdots n_k = n}} \prod_{\ell=1}^k \frac{\Lambda(n_\ell)}{n_\ell^{s_\ell} \log(n_\ell)}.$$

Then for all complex numbers  $z$  with  $\operatorname{Re}(z) > 1 - \sigma_0$  we have

$$\prod_{\ell=1}^k \log \zeta(s_\ell + z) = \sum_{n=1}^{\infty} \frac{F_{\mathbf{s}}(n)}{n^z}.$$

The main result of this subsection is the following proposition.

**Proposition 5.1.** *Let  $T$  be large,  $s_1, \dots, s_k$  be as above, and  $\mathcal{E}(T)$  be as in Lemma 5.4 below. Then, there exist positive constants  $a(\sigma_0), b(\sigma_0)$  such that if  $k \leq a(\sigma_0)(\log T)/\log \log T$  and  $t \in [T, 2T] \setminus \mathcal{E}(T)$  then*

$$\prod_{j=1}^k \log \zeta(s_j + it) = \sum_{n \leq T^{\delta/8}} \frac{F_{\mathbf{s}}(n)}{n^{it}} + O(T^{-b(\sigma_0)}).$$

This depends on a sequence of fairly standard lemmas which we now describe.

**Lemma 5.2.** *With the same notation as above, we have*

$$|F_{\mathbf{s}}(n)| \leq \frac{(2 \log n)^k}{n^{\sigma_0}}.$$

*Proof.* We have

$$|F_{\mathbf{s}}(n)| \leq \frac{1}{n^{\sigma_0} (\log 2)^k} \sum_{\substack{n_1, n_2, \dots, n_k \geq 2 \\ n_1 n_2 \cdots n_k = n}} \prod_{\ell=1}^k \Lambda(n_\ell) \leq \frac{2^k}{n^{\sigma_0}} \left( \sum_{m|n} \Lambda(m) \right)^k \leq \frac{(2 \log n)^k}{n^{\sigma_0}}.$$

□

**Lemma 5.3.** *Let  $y \geq 2$  and  $|t| \geq y + 3$  be real numbers. Suppose that the rectangle  $\{z : \sigma_0 - \delta/2 < \operatorname{Re}(z) \leq 1, |\operatorname{Im}(z) - t| \leq y + 2\}$  is free of zeros of  $\zeta(z)$ . Then, for all complex numbers  $s$  such that  $\operatorname{Re}(s) \geq \sigma_0 - \delta/4$  and  $|\operatorname{Im}(s)| \leq y$  we have*

$$\log \zeta(s + it) \ll_{\sigma_0} \log |t|.$$

*Proof.* This follows from Theorem 9.6 B of Titchmarsh. □

**Lemma 5.4.** *Let  $s_1, \dots, s_k$  be as above. Then, there exists a set  $\mathcal{E}(T) \subset [T, 2T]$  with measure  $\operatorname{meas}(\mathcal{E}(T)) \ll T^{1-\delta/8}$ , and such that for all  $t \in [T, 2T] \setminus \mathcal{E}(T)$  we have  $\zeta(s_j + it + z) \neq 0$  for every  $1 \leq j \leq k$  and every  $z$  in the rectangle  $\{z : -\delta/2 < \operatorname{Re}(z) \leq 1, |\operatorname{Im}(z)| \leq 3T^{\delta/4}\}$ .*

*Proof.* For every  $1 \leq j \leq k$ , let  $\mathcal{E}_j(T)$  be the set of  $t \in [T, 2T]$  such that the rectangle  $\{z : -\delta/2 < \operatorname{Re}(z) \leq 1, |\operatorname{Im}(z)| \leq 3T^{\delta/4}\}$  has a zero of  $\zeta(s_j + it + z)$ . Then, by the classical zero density estimate  $N(\sigma, T) \ll T^{3/2-\sigma}(\log T)^5$ , we deduce that

$$\operatorname{meas}(\mathcal{E}_j(T)) \ll T^{\delta/4} T^{3/2-\sigma_0+\delta/2} (\log T)^5 < T^{1-\delta/4} (\log T)^5.$$

We take  $\mathcal{E}(T) = \cup_{j=1}^k \mathcal{E}_j(T)$ . Then  $\mathcal{E}(T)$  satisfies the assumptions of the lemma, since  $\operatorname{meas}(\mathcal{E}(T)) \ll T^{1-\delta/4} (\log T)^6 \ll T^{1-\delta/8}$ .  $\square$

We are now ready to prove Proposition 5.1.

*Proof of Proposition 5.1.* Let  $x = \lfloor T^{\delta/8} \rfloor + 1/2$ . Let  $c = 1 - \sigma_0 + 1/\log T$ , and  $Y = T^{\delta/4}$ . Then by Perron's formula, we have for  $t \in [T, 2T] \setminus \mathcal{E}(T)$

$$\frac{1}{2\pi i} \int_{c-iY}^{c+iY} \left( \prod_{j=1}^k \log \zeta(s_j + it + z) \right) \frac{x^z}{z} dz = \sum_{n \leq x} \frac{F_s(n)}{n^{it}} + O \left( \frac{x^c}{Y} \sum_{n=1}^{\infty} \frac{|F_s(n)|}{n^c |\log(x/n)|} \right).$$

To bound the error term of this last estimate, we split the sum into three parts:  $n \leq x/2$ ,  $x/2 < n < 2x$  and  $n \geq 2x$ . The terms in the first and third parts satisfy  $|\log(x/n)| \geq \log 2$ , and hence their contribution is

$$\ll \frac{x^{1-\sigma_0}}{Y} \sum_{n=1}^{\infty} \frac{|F_s(n)|}{n^c} \leq \frac{x^{1-\sigma_0}}{Y} \left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma_0+c} \log n} \right)^k \leq \frac{x^{1-\sigma_0} (2 \log T)^k}{Y} \ll T^{-b(\sigma_0)},$$

or some positive constant  $b(\sigma_0)$ , if  $a(\sigma_0)$  is sufficiently small. To handle the contribution of the terms  $x/2 < n < 2x$ , we put  $r = x - n$ , and use that  $|\log(x/n)| \gg |r|/x$ . Then by Lemma 5.2 we deduce that the contribution of these terms is

$$\ll \frac{x^{1-\sigma_0} (3 \log x)^k}{Y} \sum_{r \leq x} \frac{1}{r} \ll \frac{x^{1-\sigma_0} (3 \log x)^{k+1}}{Y} \ll T^{-b(\sigma_0)}.$$

We now move the contour to the line  $\operatorname{Re}(s) = -\delta/4$ . By Lemma 5.4, we do not encounter any zeros of  $\zeta(s_j + it + z)$  since  $t \in [T, 2T] \setminus \mathcal{E}(T)$ . We pick up a simple pole at  $z = 0$  which leaves a residue  $\prod_{j=1}^k \log \zeta(s_j + it)$ . Also Lemma 5.3 implies that for any  $z$  on our contour we have

$$|\log \zeta(s_j + it + z)| \leq c(\sigma_0) \log T,$$

for all  $j$  where  $c(\sigma_0)$  is a positive constant. Therefore, we deduce that

$$\frac{1}{2\pi i} \int_{c-iY}^{c+iY} \left( \prod_{j=1}^k \log \zeta(s_j + it + z) \right) \frac{x^z}{z} dz = \prod_{j=1}^k \log \zeta(s_j + it) + E_1,$$

where

$$\begin{aligned} E_1 &= \frac{1}{2\pi i} \left( \int_{c-iY}^{-\delta/4-iY} + \int_{-\delta/4-iY}^{-\delta/4+iY} + \int_{-\delta/4+iY}^{c+iY} \right) \left( \prod_{j=1}^k \log \zeta(s_j + it + z) \right) \frac{x^z}{z} dz \\ &\ll \frac{x^{1-\sigma_0} (c(\sigma_0) \log T)^k}{Y} + x^{-\delta/4} (c(\sigma_0) \log T)^k \log Y \ll T^{-b(\sigma_0)}, \end{aligned}$$



as desired.  $\square$

## 5.2. An Asymptotic formula for the moment of products of shifts of $\log \zeta(s)$ .

*Proof of Theorem 2.3.* Let  $\mathcal{E}_1(T)$  and  $\mathcal{E}_2(T)$  be the corresponding exceptional sets for  $\mathbf{s}$  and  $\mathbf{r}$  respectively as in Lemma 5.4, and let  $\mathcal{E}(T) = \mathcal{E}_1(T) \cup \mathcal{E}_2(T)$ . First, note that if  $t \in [T, 2T] \setminus \mathcal{E}(T)$  then by Proposition 5.1 and Lemma 5.3 we have

$$\left| \sum_{n \leq x} \frac{F_{\mathbf{s}}(n)}{n^{it}} \right| \ll (c(\sigma_0) \log T)^k, \text{ and } \left| \sum_{m \leq x} \overline{F_{\mathbf{r}}(m)} m^{it} \right| \ll (c(\sigma_0) \log T)^\ell,$$

for some positive constant  $c(\sigma_0)$ . Let  $x = T^{(\sigma_0 - 1/2)/8}$ . Then, it follows from Proposition 5.1 that

$$\begin{aligned} (5.1) \quad & \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \prod_{j=1}^k \log \zeta(s_j + it) \right) \left( \prod_{j=1}^\ell \log \overline{\zeta(r_j + it)} \right) dt \\ &= \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \sum_{n \leq x} \frac{F_{\mathbf{s}}(n)}{n^{it}} \right) \left( \sum_{m \leq x} \overline{F_{\mathbf{r}}(m)} m^{it} \right) dt + O(T^{-b(\sigma_0)} (c(\sigma_0) \log T)^{\max(k, \ell)}) \\ &= \frac{1}{T} \int_T^{2T} \left( \sum_{n \leq x} \frac{F_{\mathbf{s}}(n)}{n^{it}} \right) \left( \sum_{m \leq x} \overline{F_{\mathbf{r}}(m)} m^{it} \right) dt + O(T^{-b(\sigma_0)/2}). \end{aligned}$$

Furthermore, we have

$$(5.2) \quad \frac{1}{T} \int_T^{2T} \left( \sum_{n \leq x} \frac{F_{\mathbf{s}}(n)}{n^{it}} \right) \left( \sum_{m \leq x} \overline{F_{\mathbf{r}}(m)} m^{it} \right) dt = \sum_{m, n \leq x} F_{\mathbf{s}}(n) \overline{F_{\mathbf{r}}(m)} \frac{1}{T} \int_T^{2T} \left( \frac{m}{n} \right)^{it} dt.$$

The contribution of the diagonal terms  $n = m$  equals  $\sum_{n \leq x} F_{\mathbf{s}}(n) \overline{F_{\mathbf{r}}(n)}$ . On the other hand, by Lemma 5.2 the contribution of the off-diagonal terms  $n \neq m$  is

$$(5.3) \quad \ll \frac{1}{T} \sum_{\substack{m, n \leq x \\ m \neq n}} \frac{(2 \log n)^k (2 \log m)^\ell}{(mn)^{\sigma_0}} \frac{1}{|\log(m/n)|} \ll \frac{x^{3-2\sigma_0} (2 \log x)^{k+\ell}}{T} \ll T^{-1/2},$$

since  $|\log(m/n)| \gg 1/x$ .

Furthermore, it follows from (2.3) that

$$(5.4) \quad \mathbb{E} \left( \prod_{j=1}^k \log \zeta(s_j, X) \right) \left( \prod_{j=1}^\ell \log \overline{\zeta(r_j, X)} \right) = \sum_{n=1}^{\infty} F_{\mathbf{s}}(n) \overline{F_{\mathbf{r}}(n)} = \sum_{n \leq x} F_{\mathbf{s}}(n) \overline{F_{\mathbf{r}}(n)} + E_2,$$

where

$$|E_2| \leq \sum_{n > x} \frac{(2 \log n)^{k+\ell}}{n^{2\sigma_0}}.$$

Since the function  $(\log t)^\beta/t^\alpha$  is decreasing for  $t > \exp(\beta/\alpha)$ , then with the choice  $\alpha = (2\sigma_0 - 1)/2$  we obtain

$$E_2 \leq \frac{(2 \log x)^{k+\ell}}{x^\alpha} \sum_{n>x} \frac{1}{n^{1+\alpha}} \ll \frac{(2 \log x)^{k+\ell}}{x^{2\alpha}} \ll x^{-\alpha}.$$

Combining this with (5.2), (5.3), and (5.4) completes the proof.  $\square$

## 6. THE CHARACTERISTIC FUNCTION OF JOINT SHIFTS OF $\log \zeta(s)$

*Proof of Theorem 2.4.* Let  $\mathcal{E}(T)$  be the exceptional set corresponding to the points  $s_1, \dots, s_J$  in Lemma 5.4. Let  $N = \lfloor c_4 \log T / (2(\log \log T)) \rfloor$  where  $c_4$  is defined in the statement of Theorem 2.3. Then,  $\Phi_T(\mathbf{u}, \mathbf{v})$  equals

$$\begin{aligned} & \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \exp \left( i \left( \sum_{j=1}^J (u_j \operatorname{Re} \log \zeta(s_j + it) + v_j \operatorname{Im} \log \zeta(s_j + it)) \right) \right) dt + O \left( T^{-\frac{\sigma-1/2}{8}} \right) \\ (6.1) \quad & = \sum_{n=0}^{2N-1} \frac{i^n}{n!} \cdot \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \sum_{j=1}^J (u_j \operatorname{Re} \log \zeta(s_j + it) + v_j \operatorname{Im} \log \zeta(s_j + it)) \right)^n dt + E_3, \end{aligned}$$

where

$$E_3 \ll T^{-\frac{\sigma-1/2}{8}} + \frac{1}{(2N)!} \left( \frac{2c_1(\log T)^\sigma}{J} \right)^{2N} \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \sum_{j=1}^J |\log \zeta(s_j + it)| \right)^{2N} dt,$$

by our assumption on the  $u_j$ 's and  $v_j$ 's where  $c_1 = c_1(\sigma)$ . Now, by Theorem 2.3 along with Lemma 4.3, we obtain that for all  $1 \leq j \leq J$

$$(6.2) \quad \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} |\log \zeta(s_j + it)|^{2N} dt \ll \mathbb{E}(|\log \zeta(s_j, X)|^{2N}) \ll \left( \frac{c_8(\sigma) N^{1-\sigma}}{(\log N)^\sigma} \right)^{2N},$$

for some positive constant  $c_8 = c_8(\sigma)$ . Furthermore, by Minkowski's inequality we have

$$\frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \sum_{j=1}^J |\log \zeta(s_j + it)| \right)^{2N} dt \ll \left( c_8 J \frac{N^{1-\sigma}}{(\log N)^\sigma} \right)^{2N}.$$

Therefore, by Stirling's formula we deduce that

$$E_3 \ll T^{-\frac{\sigma-1/2}{8}} + \left( 3c_1 c_8 \frac{(\log T)^\sigma}{(N \log N)^\sigma} \right)^{2N} \ll e^{-N},$$

if  $c_1$  is sufficiently small compared to  $c_4$  and  $c_8$ .

Next, we handle the main term of (6.1). Let  $\tilde{u}_j = (u_j + iv_j)/2$  and  $\tilde{v}_j = (u_j - iv_j)/2$ . Then, it follows from Theorem 2.3 that for all  $0 \leq n \leq 2N - 1$  we have

$$\begin{aligned}
& \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \sum_{j=1}^J (u_j \operatorname{Re} \log \zeta(s_j + it) + v_j \operatorname{Im} \log \zeta(s_j + it)) \right)^n dt \\
& \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \left( \sum_{j=1}^J (\tilde{u}_j \log \zeta(s_j + it) + \tilde{v}_j \log \overline{\zeta(s_j + it)}) \right)^n dt \\
& = \sum_{\substack{k_1, \dots, k_{2J} \geq 0 \\ k_1 + \dots + k_{2J} = n}} \binom{n}{k_1, k_2, \dots, k_{2J}} \prod_{j=1}^J \tilde{u}_j^{k_j} \prod_{\ell=1}^J \tilde{v}_\ell^{k_{J+\ell}} \\
& \quad \times \frac{1}{T} \int_{[T, 2T] \setminus \mathcal{E}(T)} \prod_{j=1}^J (\log \zeta(s_j + it))^{k_j} \prod_{\ell=1}^J (\log \overline{\zeta(s_\ell + it)})^{k_{J+\ell}} dt \\
& = \sum_{\substack{k_1, \dots, k_{2J} \geq 0 \\ k_1 + \dots + k_{2J} = n}} \binom{n}{k_1, k_2, \dots, k_{2J}} \prod_{j=1}^J \tilde{u}_j^{k_j} \prod_{\ell=1}^J \tilde{v}_\ell^{k_{J+\ell}} \\
& \quad \times \mathbb{E} \left( \prod_{j=1}^J (\log \zeta(s_j, X))^{k_j} \prod_{\ell=1}^J (\log \overline{\zeta(s_\ell, X)})^{k_{J+\ell}} \right) + O\left(T^{-c_5} (2c_1 (\log T)^\sigma)^n\right), \\
& = \mathbb{E} \left( \left( \sum_{j=1}^J (u_j \operatorname{Re} \log \zeta(s_j, X) + v_j \operatorname{Im} \log \zeta(s_j, X)) \right)^n \right) + O\left(T^{-c_5} (2c_1 (\log T)^\sigma)^n\right).
\end{aligned}$$

Inserting this estimate in (6.1), we derive

$$\begin{aligned}
\Phi_T(\mathbf{u}, \mathbf{v}) &= \sum_{n=0}^{2N-1} \frac{i^n}{n!} \mathbb{E} \left( \left( \sum_{j=1}^J (u_j \operatorname{Re} \log \zeta(s_j, X) + v_j \operatorname{Im} \log \zeta(s_j, X)) \right)^n \right) + O(e^{-N}) \\
&= \Phi_{\text{rand}}(\mathbf{u}, \mathbf{v}) + E_4.
\end{aligned}$$

where

$$E_4 \ll e^{-N} + \frac{1}{(2N)!} \left( \frac{2c_1 (\log T)^\sigma}{J} \right)^{2N} \mathbb{E} \left( \left( \sum_{j=1}^J |\log \zeta(s_j, X)| \right)^{2N} \right) \ll e^{-N}$$

by (6.2) and Minkowski's inequality. This completes the proof.  $\square$

## 7. DISCREPANCY ESTIMATES FOR THE DISTRIBUTION OF SHIFTS

The deduction of Theorem 2.5 from Theorem 2.4 uses Beurling-Selberg functions. For  $z \in \mathbb{C}$  let

$$H(z) = \left( \frac{\sin \pi z}{\pi} \right)^2 \left( \sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2} + \frac{2}{z} \right) \quad \text{and} \quad K(z) = \left( \frac{\sin \pi z}{\pi z} \right)^2.$$

Beurling proved that the function  $B^+(x) = H(x) + K(x)$  majorizes  $\operatorname{sgn}(x)$  and its Fourier transform has restricted support in  $(-1, 1)$ . Similarly, the function  $B^-(x) =$

$H(x) - K(x)$  minorizes  $\text{sgn}(x)$  and its Fourier transform has the same property (see Vaaler [13] Lemma 5).

Let  $\Delta > 0$  and  $a, b$  be real numbers with  $a < b$ . Take  $\mathcal{I} = [a, b]$  and define

$$F_{\mathcal{I}}(z) = \frac{1}{2} \left( B^-(\Delta(z - a)) + B^-(\Delta(b - z)) \right).$$

The function  $F_{\mathcal{I}}$  has the following remarkable properties. First, we have

$$(7.1) \quad 0 \leq \mathbf{1}_{\mathcal{I}}(x) - F_{\mathcal{I}}(x) \leq K(\Delta(x - a)) + K(\Delta(b - x)),$$

for all  $x \in \mathbb{R}$ . For  $x \neq a, b$ , this follows from the identity

$$\mathbf{1}_{\mathcal{I}}(x) = \frac{\text{sgn}(\Delta(x - a)) + \text{sgn}(\Delta(b - x))}{2}$$

together with the inequality  $B^-(x) \leq \text{sgn}(x) \leq B^+(x)$ . Since  $H(x)$  and  $K(x)$  are continuous, the restrictions on  $x$  can be removed.

Additionally, one has

$$(7.2) \quad \widehat{F}_{\mathcal{I}}(\xi) = \begin{cases} \widehat{\mathbf{1}}_{\mathcal{I}}(\xi) + O\left(\frac{1}{\Delta}\right) & \text{if } |\xi| < \Delta, \\ 0 & \text{if } |\xi| \geq \Delta. \end{cases}$$

The first estimate above follows from (7.1) and the second follows from the fact that the Fourier transform of  $B^-$  is supported in  $(-1, 1)$ . Before proving Theorem 2.5 we first require the following lemmas.

**Lemma 7.1.** *For  $x \in \mathbb{R}$  we have  $|F_{\mathcal{I}}(x)| \leq 1$ .*

*Proof.* It suffices to prove the lemma for  $\Delta = 1$ . Also, note that we only need to show that  $F_{\mathcal{I}}(x) \geq -1$ . From the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n - z)^2} = \left( \frac{\pi}{\sin \pi z} \right)^2$$

it follows that for  $y \geq 0$

$$(7.3) \quad H(y) = 1 - K(y)G(y)$$

where

$$G(y) = 2y^2 \sum_{m=0}^{\infty} \frac{1}{(y + m)^2} - 2y - 1.$$

In Lemma 5 of [13], Vaaler shows for  $y \geq 0$  that

$$(7.4) \quad 0 \leq G(y) \leq 1.$$

Also, note that for each  $m \geq 1$ , and  $0 < y \leq 1$  one has  $\frac{m}{(y+m)^3} \leq \int_{m-1}^m \frac{t}{(y+t)^3} dt$  so that for  $0 < y \leq 1$

$$(7.5) \quad G'(y) = 4y \sum_{m \geq 1} \frac{m}{(y + m)^3} - 2 \leq 4y \int_0^{\infty} \frac{t}{(y + t)^3} dt - 2 = 0.$$

First consider the case  $a \leq x \leq b$ . By (7.3) we get that in this range

$$F_{\mathcal{I}}(x) = \frac{1}{2} (2 - K(x-a)(G(x-a)+1) - K(b-x)(G(b-x)+1)),$$

which along with (7.4) implies  $F_{\mathcal{I}}(x) \geq -1$  for  $a \leq x \leq b$ . Now consider the case  $x < a$ . Since  $H$  is an odd function (7.3) and (7.4) imply

$$\begin{aligned} F_{\mathcal{I}}(x) &= \frac{1}{2} (K(x-a)(G(a-x)-1) - K(b-x)(G(b-x)+1)) \\ &\geq \frac{1}{2} (-K(x-a) - 2K(x-b)), \end{aligned}$$

which is  $\geq -1$  if  $K(x-b) \leq 1/2$ . If  $K(x-b) \geq 1/2$  we also have  $K(x-a) > K(x-b)$  and  $0 < b-x < 1$ . By this and (7.5) we have in this range as well that

$$F_{\mathcal{I}}(x) \geq \frac{1}{2} (K(x-b)(G(a-x) - G(b-x) - 2)) \geq -1.$$

Hence,  $F_{\mathcal{I}}(x) \geq -1$  for  $x < a$ . The remaining case when  $x > b$  follows from a similar argument.  $\square$

**Lemma 7.2.** *Fix  $1/2 < \sigma < 1$ , and let  $s$  be a complex number such that  $\operatorname{Re}(s) = \sigma$  and  $|\operatorname{Im}(s)| \leq T^{\frac{1}{4}(\sigma - \frac{1}{2})}$ . Then there exists a positive constant  $c_1(\sigma)$  such that for  $|u| \leq c_1(\sigma)(\log T)^\sigma$  we have*

$$\Phi_T(u, 0) \ll \exp\left(\frac{-u}{5 \log u}\right) \quad \text{and} \quad \Phi_T(0, u) \ll \exp\left(\frac{-u}{5 \log u}\right).$$

*Proof.* By a straightforward modification of Lemma 6.3 of [10] one has that

$$\mathbb{E}\left(\exp\left(iu \operatorname{Re} \log \zeta(s, X)\right)\right) \ll \exp\left(-\frac{u}{5 \log u}\right),$$

and

$$\mathbb{E}\left(\exp\left(iu \operatorname{Im} \log \zeta(s, X)\right)\right) \ll \exp\left(-\frac{u}{5 \log u}\right).$$

Using the first bound and applying Theorem 2.4 with  $J = 1$  establishes the first claim. The second claim follows similarly by using the second bound and Theorem 2.4.  $\square$

*Proof of Theorem 2.5.* First, we claim that it suffices to estimate the discrepancy over  $(\mathcal{R}_1, \dots, \mathcal{R}_J)$  such that for each  $j$  we have  $\mathcal{R}_j \subset [-\sqrt{\log T}, \sqrt{\log T}] \times [-\sqrt{\log T}, \sqrt{\log T}]$ . To see this consider  $(\widetilde{\mathcal{R}}_1, \dots, \widetilde{\mathcal{R}}_J)$  where  $\widetilde{\mathcal{R}}_j = \mathcal{R}_j \cap [-\sqrt{\log T}, \sqrt{\log T}] \times [-\sqrt{\log T}, \sqrt{\log T}]$ . It follows that

$$\begin{aligned} &\left| \mathbb{P}_T\left(\log \zeta(s_j + it) \in \mathcal{R}_j, \forall j \leq J\right) - \mathbb{P}_T\left(\log \zeta(s_1 + it) \in \widetilde{\mathcal{R}}_1, \log \zeta(s_j + it) \in \mathcal{R}_j, 2 \leq j \leq J\right) \right| \\ &\ll \mathbb{P}_T\left(|\log \zeta(s_1 + it)| \geq \sqrt{\log T}\right) \ll \exp\left(-\sqrt{\log T}\right), \end{aligned}$$

where the last bound follows from Theorem 1.1 and Remark 1 of [9]. Repeating this argument gives

$$\left| \mathbb{P}_T \left( \log \zeta(s_j + it) \in \widetilde{\mathcal{R}}_j, \forall j \leq J \right) - \mathbb{P}_T \left( \log \zeta(s_j + it) \in \mathcal{R}_j, \forall j \leq J \right) \right| \ll J \exp \left( -\sqrt{\log T} \right).$$

Similarly,

$$\left| \mathbb{P} \left( \log \zeta(s_j, X) \in \widetilde{\mathcal{R}}_j, \forall j \leq J \right) - \mathbb{P} \left( \log \zeta(s_j, X) \in \mathcal{R}_j, \forall j \leq J \right) \right| \ll J \exp \left( -\sqrt{\log T} \right).$$

Hence, the error from restricting to  $(\widetilde{\mathcal{R}}_1, \dots, \widetilde{\mathcal{R}}_J)$  is negligible and establishes the claim.

Let  $\Delta = c_1(\sigma)(\log T)^\sigma/J$  and  $\mathcal{R}_j = [a_j, b_j] \times [c_j, d_j]$  for  $j = 1, \dots, J$ , with  $|b_j - a_j|, |d_j - c_j| \leq 2\sqrt{\log T}$ . Also, write  $\mathcal{I}_j = [a_j, b_j]$  and  $\mathcal{J}_j = [c_j, d_j]$ . By Fourier inversion, (7.2), and Theorem 2.4 we have that

$$\begin{aligned} (7.6) \quad & \frac{1}{T} \int_T^{2T} \prod_{j=1}^J F_{\mathcal{I}_j} \left( \operatorname{Re} \log \zeta(s_j + it) \right) F_{\mathcal{J}_j} \left( \operatorname{Im} \log \zeta(s_j + it) \right) dt \\ &= \int_{\mathbb{R}^{2J}} \left( \prod_{j=1}^J \widehat{F}_{\mathcal{I}_j}(u_j) \widehat{F}_{\mathcal{J}_j}(v_j) \right) \Phi_T(\mathbf{u}, \mathbf{v}) d\mathbf{u} d\mathbf{v} \\ &= \int_{\substack{|u_j|, |v_j| \leq \Delta \\ j=1, 2, \dots, J}} \left( \prod_{j=1}^J \widehat{F}_{\mathcal{I}_j}(u_j) \widehat{F}_{\mathcal{J}_j}(v_j) \right) \Phi_{\text{rand}}(\mathbf{u}, \mathbf{v}) d\mathbf{u} d\mathbf{v} + O \left( \left( 2\Delta \sqrt{\log T} \right)^{2J} \exp \left( -\frac{c_2 \log T}{\log \log T} \right) \right) \\ &= \mathbb{E} \left( \prod_{j=1}^J F_{\mathcal{I}_j} \left( \operatorname{Re} \log \zeta(s_j, X) \right) F_{\mathcal{J}_j} \left( \operatorname{Im} \log \zeta(s_j, X) \right) \right) + O \left( \exp \left( -\frac{c_2 \log T}{2 \log \log T} \right) \right). \end{aligned}$$

Next note that  $\widehat{K}(\xi) = \max(0, 1 - |\xi|)$ . Applying Fourier inversion, Theorem 2.4 with  $J = 1$ , and Lemma 7.2 we have that

$$\frac{1}{T} \int_T^{2T} K \left( \Delta \cdot \left( \operatorname{Re} \log \zeta(s + it) - \alpha \right) \right) dt = \frac{1}{\Delta} \int_{-\Delta}^{\Delta} \left( 1 - \frac{|\xi|}{\Delta} \right) e^{-2\pi i \alpha \xi} \Phi_T(\xi, 0) d\xi \ll \frac{1}{\Delta},$$

where  $\alpha$  is an arbitrary real number and  $s \in \mathbb{C}$  satisfies  $\sigma \leq \operatorname{Re}(s) < 1$  and  $|\operatorname{Im}(s)| < T^{\frac{1}{4}(\sigma - \frac{1}{2})}$ . By this and (7.1) we get that

$$(7.7) \quad \frac{1}{T} \int_T^{2T} F_{\mathcal{I}_1} \left( \operatorname{Re} \log \zeta(s_1 + it) \right) dt = \frac{1}{T} \int_T^{2T} \mathbf{1}_{\mathcal{I}_1} \left( \operatorname{Re} \log \zeta(s_1 + it) \right) dt + O(1/\Delta).$$

Lemma 7.1 implies that  $|F_{\mathcal{I}_j}(x)|, |F_{\mathcal{J}_j}(x)| \leq 1$  for  $j = 1, \dots, J$ . Hence, by this and (7.7)

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \prod_{j=1}^J F_{\mathcal{I}_j} \left( \operatorname{Re} \log \zeta(s_j + it) \right) F_{\mathcal{J}_j} \left( \operatorname{Im} \log \zeta(s_j + it) \right) dt \\ &= \frac{1}{T} \int_T^{2T} \mathbf{1}_{\mathcal{I}_1} \left( \operatorname{Re} \log \zeta(s_1 + it) \right) F_{\mathcal{J}_1} \left( \operatorname{Im} \log \zeta(s_1 + it) \right) \\ & \quad \times \prod_{j=2}^J F_{\mathcal{I}_j} \left( \operatorname{Re} \log \zeta(s_j + it) \right) F_{\mathcal{J}_j} \left( \operatorname{Im} \log \zeta(s_j + it) \right) dt + O(1/\Delta). \end{aligned}$$

Iterating this argument and using an analog of (7.7) for  $\operatorname{Im} \log \zeta(s + it)$ , which is proved in the same way, gives

$$\begin{aligned} (7.8) \quad & \frac{1}{T} \int_T^{2T} \prod_{j=1}^J F_{\mathcal{I}_j} \left( \operatorname{Re} \log \zeta(s_j + it) \right) F_{\mathcal{J}_j} \left( \operatorname{Im} \log \zeta(s_j + it) \right) dt \\ &= \mathbb{P}_T \left( \log \zeta(s_j + it) \in \mathcal{R}_j, \forall j \leq J \right) + O \left( \frac{J}{\Delta} \right). \end{aligned}$$

Similarly, it can be shown that

$$\begin{aligned} (7.9) \quad & \mathbb{E} \left( \prod_{j=1}^J F_{\mathcal{I}_j} \left( \operatorname{Re} \log \zeta(s_j, X) \right) F_{\mathcal{J}_j} \left( \operatorname{Im} \log \zeta(s_j, X) \right) \right) = \mathbb{P} \left( \log \zeta(s_j, X) \in \mathcal{R}_j, \forall j \leq J \right) \\ & \quad + O \left( \frac{J}{\Delta} \right). \end{aligned}$$

Using (7.8) and (7.9) in (7.6) completes the proof. □

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